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Frequency methods for solving nonlinear systems

Fourier series

$f(x)$ periodic function with period 2π ... $f(x + 2\pi) = f(x)$

$$f(x) = \frac{b_0}{2} + \sum_{n=1}^{\infty} \left(a_n \sin nx + b_n \cos nx \right)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \quad \ldots \quad n = 1,2,3,\ldots$$

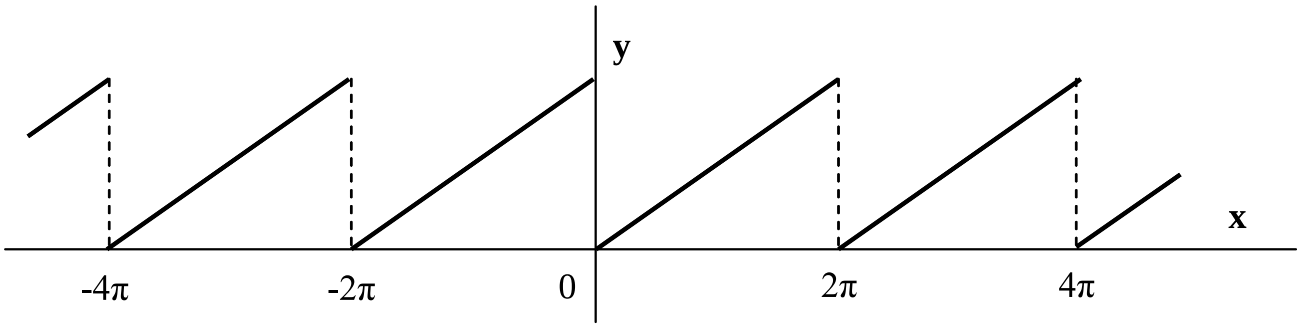
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \quad \ldots \quad n = 0,1,2,\ldots$$

base interval is arbitrary interval $\langle a, a + 2\pi \rangle$

(not only $\langle -\pi, \pi \rangle$ but also often $\langle 0, 2\pi \rangle$ ... then $\int_0^{2\pi} \ldots$)

Example: Let function $f(x)$ is periodic with period 2π and let it is in base interval $\langle 0, 2\pi \rangle$ is given by equation $f(x) = x$.

Transform it to Fourier series.



$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\sin nx dx = -\frac{1}{\pi}\left[x\frac{\cos nx}{n}\right]_0^{2\pi} + \frac{1}{n\pi}\int_0^{2\pi}\cos nx\, dx = -\frac{2}{n} \quad \dots \quad n = 1,2,3,\dots$$

$$b_0 = \frac{1}{\pi}\int_0^{2\pi} x dx = 2\pi$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x\cos nx dx = -\frac{1}{\pi}\left[x\frac{\sin nx}{n}\right]_0^{2\pi} - \frac{1}{n\pi}\int_0^{2\pi}\sin nx\, dx = 0 \quad \dots \quad n = 1,2,3,\dots$$

Result

$$f(x)=\pi-2\left(\frac{\sin x}{1}+\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+\dots\right)$$

Periodic function with period 2*l*

$$f(x)=\frac{b_0}{2}+\sum_{n=1}^{\infty}\left(a_n\sin\frac{n\pi x}{l}+b_n\cos\frac{n\pi x}{l}\right)$$

$$a_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{n\pi x}{l}dx \quad \dots \quad n=1,2,3,\dots \qquad b_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}dx \quad \dots \quad n=0,1,2,\dots$$

Periodic functions with period *T*

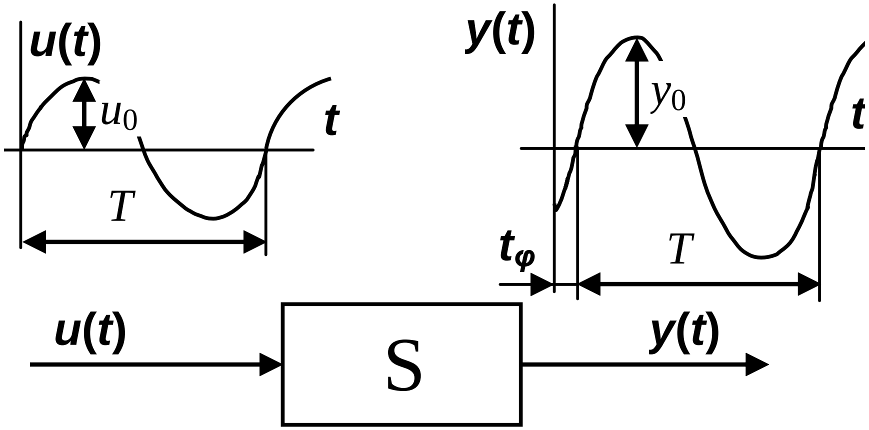
$$t\to x \qquad \frac{\pi}{l}=\frac{\pi}{T/2}=\frac{2\pi}{T}=\omega \qquad \omega\to\frac{\pi}{l}$$

$$f(t)=\frac{b_0}{2}+\sum_{n=1}^{\infty}\left(a_n\sin n\omega t+b_n\cos n\omega t\right)$$

$$a_n=\frac{2}{T}\int_{0}^{T}f(t)\sin n\omega t\,dt \quad \dots \quad n=1,2,3,\dots \qquad b_n=\frac{2}{T}\int_{0}^{T}f(t)\cos n\omega t\,dt \quad \dots \quad n=0,1,2,\dots$$

Equivalent transform

Linear systems



$$u(t) = u_0 \sin \omega t$$

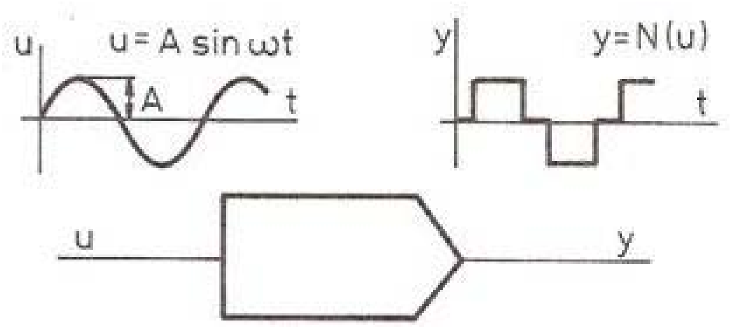
$$y(t) = y_0 \sin(\omega t + \varphi)$$

$$\mathbf{u}(t) = u_0 e^{j\omega t}$$

$$\mathbf{y}(t) = y_0 e^{j(\omega t + \varphi)}$$

$$G(j\omega) = \frac{\mathbf{y}(t)}{\mathbf{u}(t)} = \frac{y_0 e^{j(\omega t + \varphi)}}{u_0 e^{j(\omega t)}} = \frac{y_0}{u_0} e^{j\varphi}$$

Nonlinear systems – brief summary

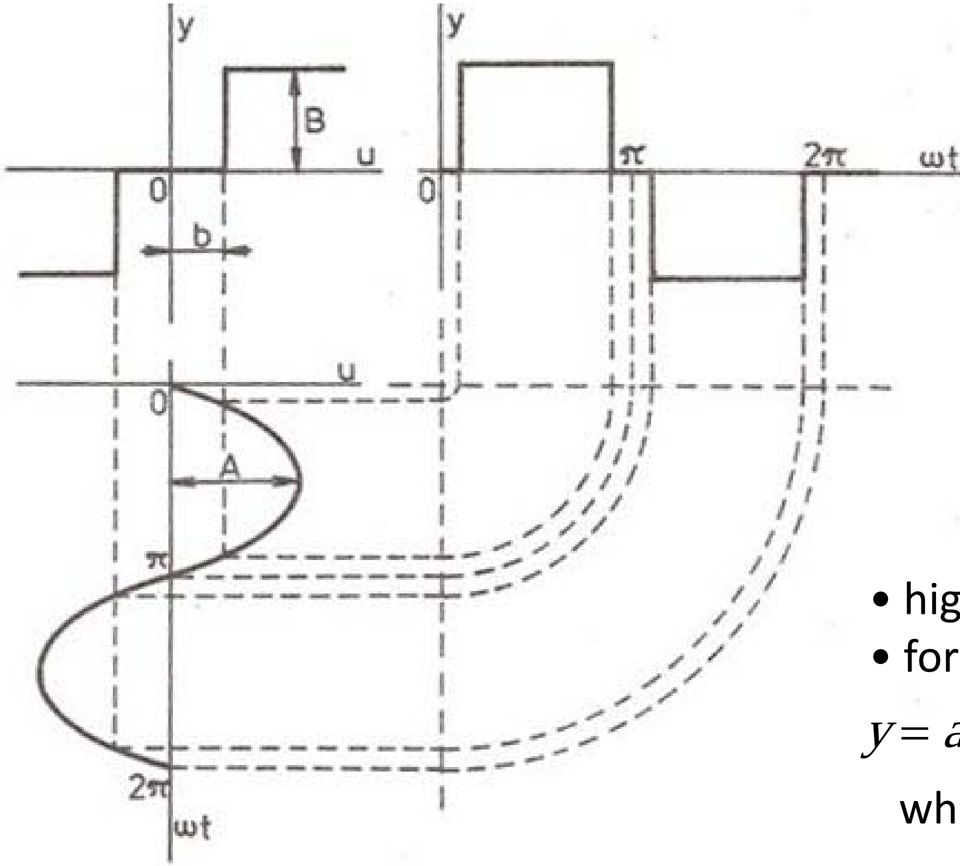


$$G_N(A) = \frac{\text{first harmonics of output}}{\text{sinusoidal input signal}} = \frac{Y}{U}$$

$$G_N(A) = a(A) + jb(A)$$

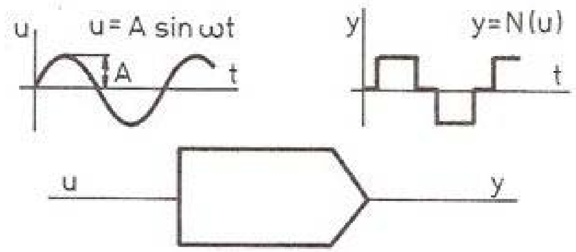
$a(A), \quad b(A)$ coefficients of Fourier series for first harmonics

$b(A)$ proportional to surface of static characteristics – for explicit nonlinerarity = 0



$$u = A\sin\omega t$$

$$y = N(u)$$



$$y = a_1 \sin\omega t + a_2 \sin\omega t + \ldots$$

$$+ \frac{b_0}{2} + b_1 \cos\omega t + b_2 \cos\omega t + \ldots$$

- higher harmonics filtered by linear parts
- for even functions is $b_0 = 0$

$$y = a_1 \sin\omega t + b_1 \cos\omega t = Aa \sin\omega t + Ab\cos\omega t$$

where $a_1 = Aa\,;\quad b_1 = Ab$

$$a_1 = \frac{2}{T}\int_0^T y(t)\sin n\omega t\,dt \qquad a = \frac{a_1}{A} = \frac{2}{TA}\int_0^T y(t)\sin n\omega t\,dt = \frac{\omega}{\pi A}\int_0^T y(t)\sin n\omega t\,dt$$

$$b_1 = \frac{2}{T}\int_0^T y(t)\cos n\omega t\,dt \qquad b = \frac{b_1}{A} = \frac{2}{TA}\int_0^T y(t)\cos n\omega t\,dt = \frac{\omega}{\pi A}\int_0^T y(t)\cos n\omega t\,dt$$

Transforming *y* to function only using sinus: $a = c\cos\varphi \quad b = c\sin\varphi$

$$c^2 = a^2 + b^2 \quad ; \quad tg\,\varphi = \frac{b}{a}$$

$$y = A(a\sin\omega t + b\cos\omega t) = A[c\cos\varphi\sin\omega t + c\sin\varphi\cos\omega t] = Ac\sin(\omega t + \varphi)$$

Equivalent transfer – definition: $G_N(A) = \frac{\text{first harmonics of output}}{\text{sinusoidal input signal}} = \frac{Y}{U}$

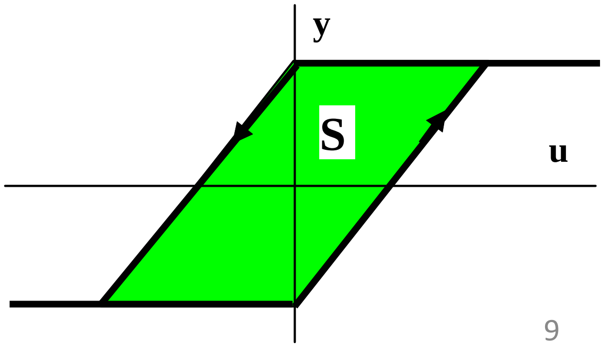
$$G_N(A) = \frac{Y}{U} = \frac{Ace^{j(\omega t+\varphi)}}{Ae^{j\omega t}} = ce^{j\varphi} = c(\cos\varphi + j\sin\varphi) = a(A) + jb(A)$$

$G_N(A)$ is independent on frequency *ω* , it is only function of input amplitude *A* !

Important:

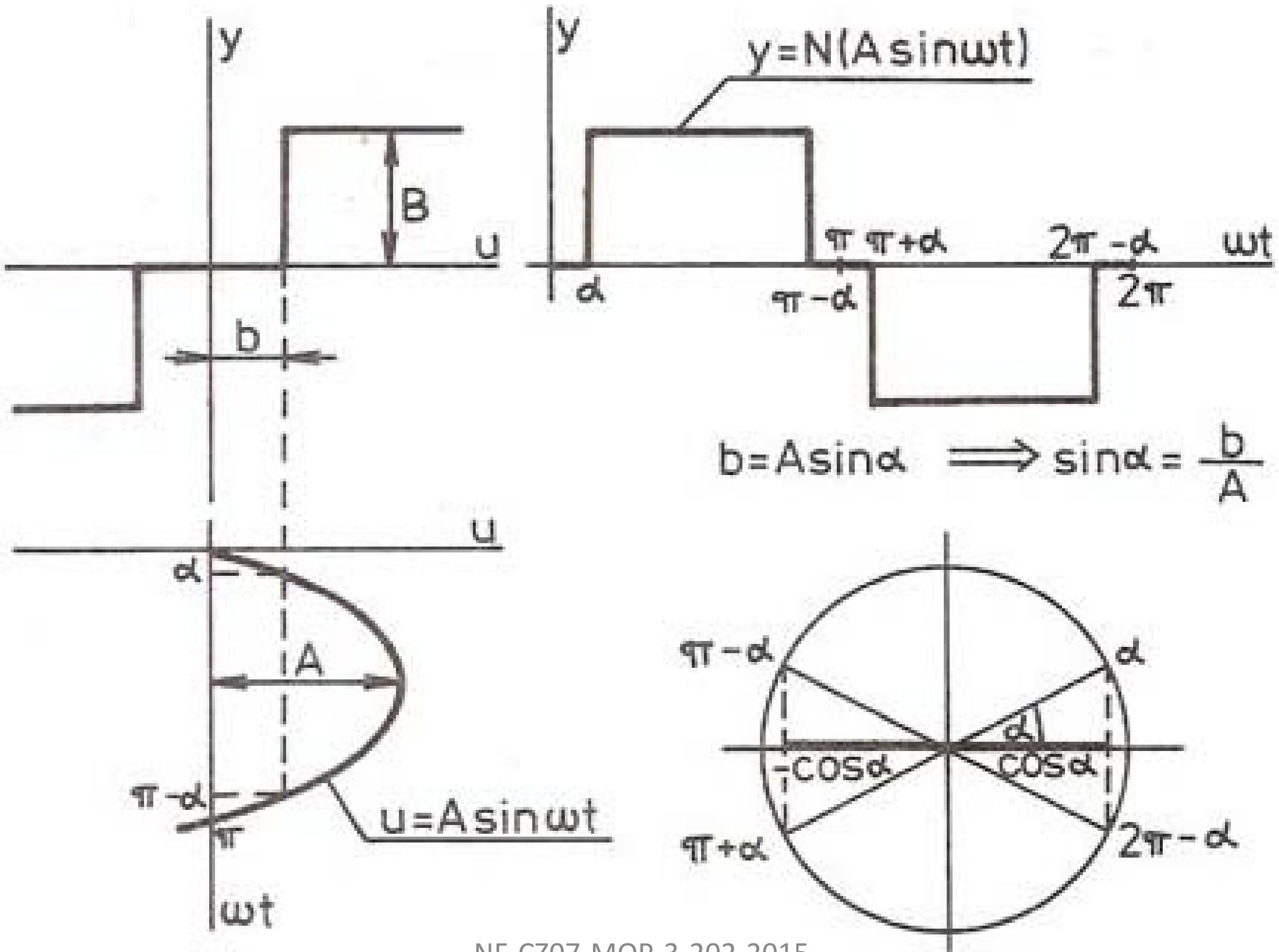
Imaginary part $b(A)$ is proportional to area of static characteristics of nonlinear member. For terms with ambiguous nonlinearity:

$$b(A) \approx S$$



$$S = 0 \quad \rightarrow \quad G_N(A) = a(A)$$

Example: Determine equivalent transfer of nonlinear control member given by statistical characteristic of relay type with insensitivity band (given values *b*, B) by figure.



Solution:

$$G_N(A) = a(A) = \frac{\omega}{\pi A}\int_0^T y(t)\sin\omega t\,dt =$$

$$= \frac{\omega}{\pi A}\left[\int_0^{\frac{\alpha}{\omega}} 0.\sin\omega t\,dt + \int_{\frac{\alpha}{\omega}}^{\frac{\pi-\alpha}{\omega}} B.\sin\omega t\,dt + \int_{\frac{\pi-\alpha}{\omega}}^{\frac{\pi+\alpha}{\omega}} 0.\sin\omega t\,dt + \int_{\frac{\pi+\alpha}{\omega}}^{\frac{2\pi-\alpha}{\omega}} (-B).\sin\omega t\,dt + \int_{\frac{2\pi-\alpha}{\omega}}^{\frac{2\pi}{\omega}} 0.\sin\omega t\,dt\right] =$$

$$\frac{\omega}{\pi A}\left[\int_{\frac{\alpha}{\omega}}^{\frac{\pi-\alpha}{\omega}} B.\sin\omega t\,dt - \int_{\frac{\pi+\alpha}{\omega}}^{\frac{2\pi-\alpha}{\omega}} B.\sin\omega t\,dt\right] = \frac{\omega B}{\pi A}\left[\left[-\frac{1}{\omega}\cos\omega t\right]_{\frac{\alpha}{\omega}}^{\frac{\pi-\alpha}{\omega}} - \left[-\frac{1}{\omega}\cos\omega t\right]_{\frac{\pi+\alpha}{\omega}}^{\frac{2\pi-\alpha}{\omega}}\right] =$$

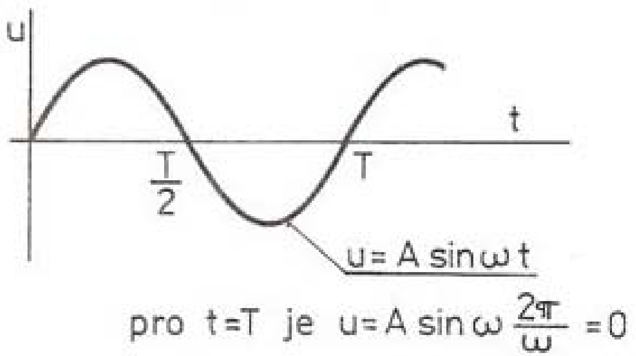
$$= \frac{B}{\pi A}\left[\left[-\cos\omega t\right]_{\frac{\alpha}{\omega}}^{\frac{\pi-\alpha}{\omega}} + \left[\cos\omega t\right]_{\frac{\pi+\alpha}{\omega}}^{\frac{2\pi-\alpha}{\omega}}\right] = \frac{B}{\pi A}\left(\left[\cos\alpha + \cos\alpha\right] + \left[\cos\alpha + \cos\alpha\right]\right) = \frac{4B}{\pi A}\cos\alpha$$

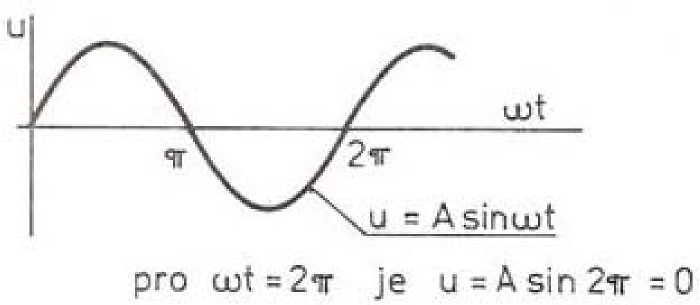
$$\cos\alpha = \sqrt{1-\sin^2\alpha}$$

$$\boxed{G_N(A) = \frac{4B}{\pi A}\sqrt{1-\frac{b^2}{A^2}}}$$

Has been used (figure)

$$\cos(\pi - \alpha) = -\cos\alpha \quad ; \quad \cos(\pi + \alpha) = -\cos\alpha \quad ; \quad \cos(2\pi - \alpha) = \cos\alpha$$





Is angle α unknown – undefined?

$$u = A\sin\omega t$$

for $\omega t = \alpha$ is $u = b \;\rightarrow\; b = A\sin\alpha$

$$\alpha = \arcsin\frac{b}{A}$$

$$G_N(A) = \frac{\omega}{\pi A}\int_0^T y(t)\sin\omega t\, dt$$

It is more appropriate to have ωt instead of t on horizontal axis.

$$G_N(A) = \frac{\omega}{\pi A}\int_0^T y(t)\sin\omega t\, dt$$

$\langle \alpha, \pi - \alpha \rangle$

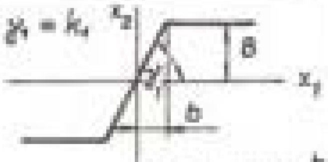
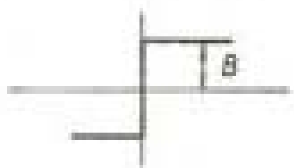
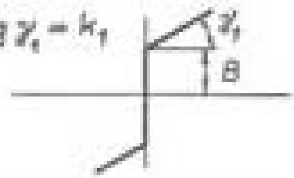
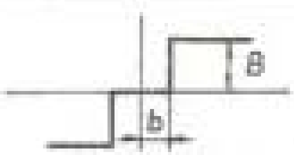
It is integrated for whole period *T*. Period is divided to segments with constant *y*(*t*). Output is ωt and therefore are intervals $\langle 0, \alpha \rangle$, $\langle \alpha, \pi - \alpha \rangle$, $\langle \pi - \alpha, \pi + \alpha \rangle$ etc. Corresponding angle must be divided by *ω*. After integration we get terms like:

$$[\cos \omega t]_{\frac{\alpha}{\omega}}^{\frac{\pi-\alpha}{\omega}}$$

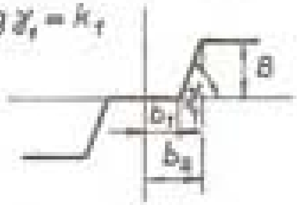
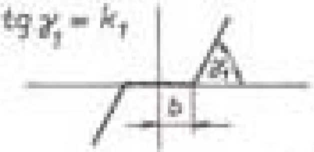
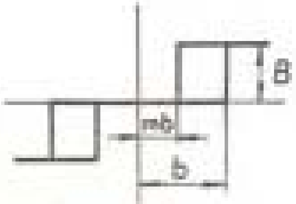
and during assignment of limits, the *ω* is eliminated.

Equivalent transfers are in the table.

Ekvivalentní přenosy — Tab. 1.3

$tg\,\gamma_1 = k_1$  $\alpha = \arcsin \frac{b}{A}$	$a(A) = \frac{k_1}{\pi}(2\alpha - \sin 2\alpha) + \frac{4B}{\pi A}\cos\alpha$
	$a(A) = \frac{4B}{\pi A}$
$tg\,\gamma_1 = k_1$ 	$a(A) = k_1 + \frac{4B}{\pi A}$
 $\alpha = \arcsin \frac{b}{A}$	$a(A) = \frac{4B}{\pi A}\cos\alpha$

Equivalent transfers are in the table.

$tg\,\gamma_1 = k_1$  $\alpha_1 = \arcsin \frac{b_1}{A}$; $\alpha_2 = \arcsin \frac{b_2}{A}$	$a(A) = \frac{k_1}{\pi A}\Big[2A(-\alpha_1 + \alpha_2) + A(\sin 2\alpha_1 - \sin 2\alpha_2) + 4b_1(\cos\alpha_2 - \cos\alpha_1) + \frac{4B}{\pi A}\cos\alpha_2\Big]$
$tg\,\gamma_1 = k_1$  $\alpha = \arcsin \frac{b}{A}$	$a(A) = \frac{k_1}{\pi A}\Big[A(\pi - 2\alpha) + A\sin 2\alpha - 4b\cos\alpha\Big]$
 $\alpha_1 = \arcsin \frac{b}{A}$; $\alpha_2 = \arcsin \frac{mb}{A}$	$a(A) = \frac{2B}{\pi A}(\cos\alpha_1 + \cos\alpha_2)$ $b(A) = -\frac{2B}{\pi A}(\sin\alpha_1 - \sin\alpha_2)$

Computation and display of equivalent transfers

Example: Nonlinear member is denoted by equation of its static characteristics

$$y = u^3$$

Compute it's equivalent transfer and amplitude characteristics in complex plane.

Solution: Input sinusoidal signal $u = A\sin\omega t$ → output signal

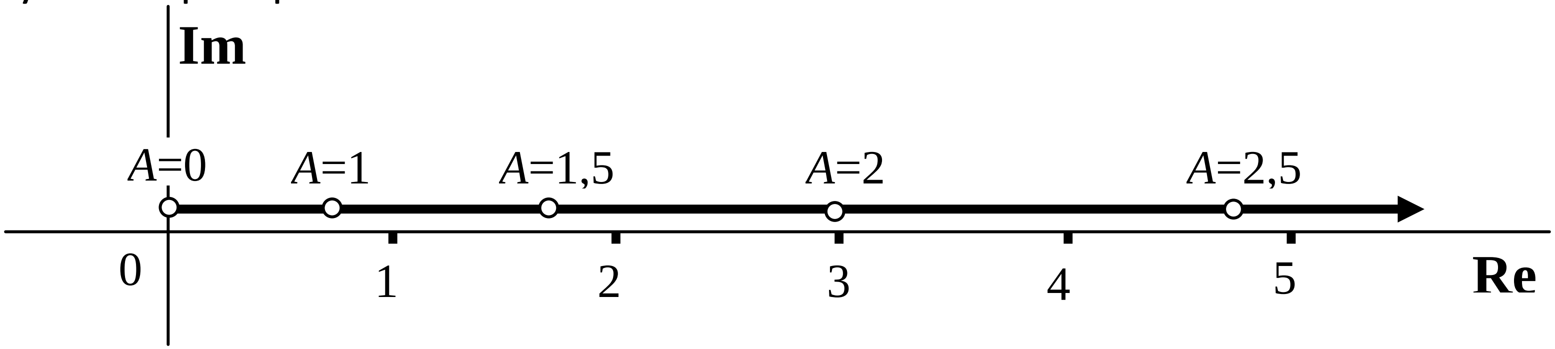
$$y = A^3\sin^3\omega t = \frac{A^3}{4}\left(3\sin\omega t - \sin 3\omega t\right)$$

Use goniometric formula $\sin^3\alpha = \frac{1}{4}\left(3\sin\alpha - \sin 3\alpha\right)$

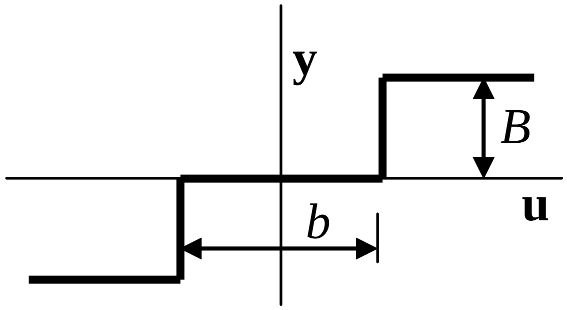
Let's ignore third harmonics – definition of equivalent transfer in complex plane

$$G_N(A) = \frac{\text{first hramonics of input}}{\text{sinusoidal output signal}} = \frac{A^3\frac{3}{4}\sin\omega t}{A\sin\omega t} = \frac{3}{4}A^3$$

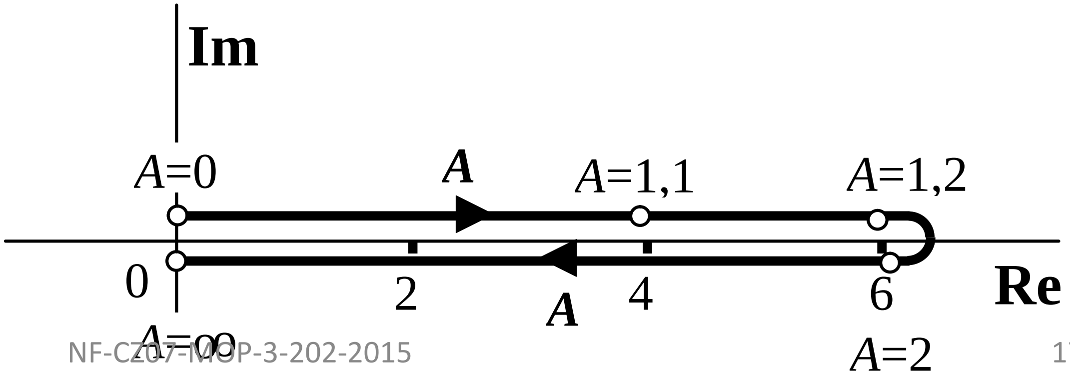
Display in complex plane



Relay with insensivity band



$$G_N(A) = \frac{4B}{\pi A}\sqrt{1 - \frac{b^2}{A^2}}$$



Determining autooscillations in nonlinear circuits

Autooscillalation (stable limit cycle)

⇩

oscillations with constant amplitude

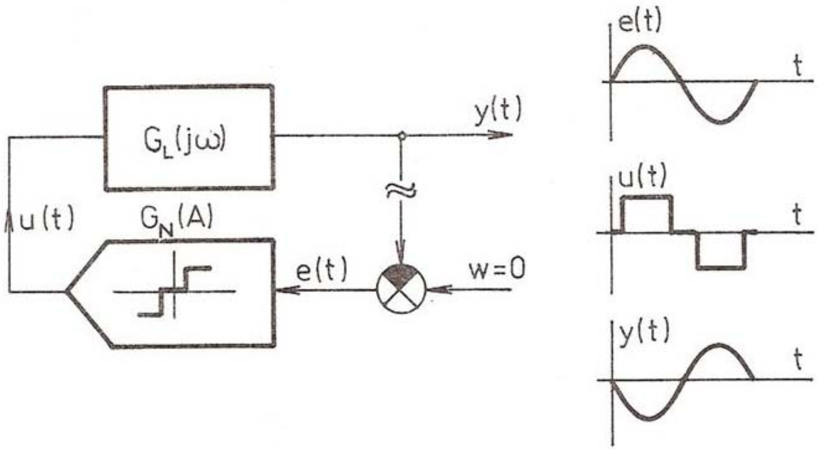
⇩

determine amplitude and frequency

⇩

avoid their creation

Existence of autooscillations → output *y*(*t*) with phase shift to input by 180° but same amplitude.



$$G_L(j\omega)G_N(A) = -1$$

common form

$$G_L(j\omega)G_N(A) + 1 = 0$$

Solution of this equation : *A* , *ω* (if the autooscillations exist).

Analytic solution:

$$\mathrm{Re}\left[G_L(j\omega)G_N(A)+1\right] = 0$$

$$\mathrm{Im}\left[G_L(j\omega)G_N(A)+1\right] = 0$$

two equations for two variables *A* , *ω*

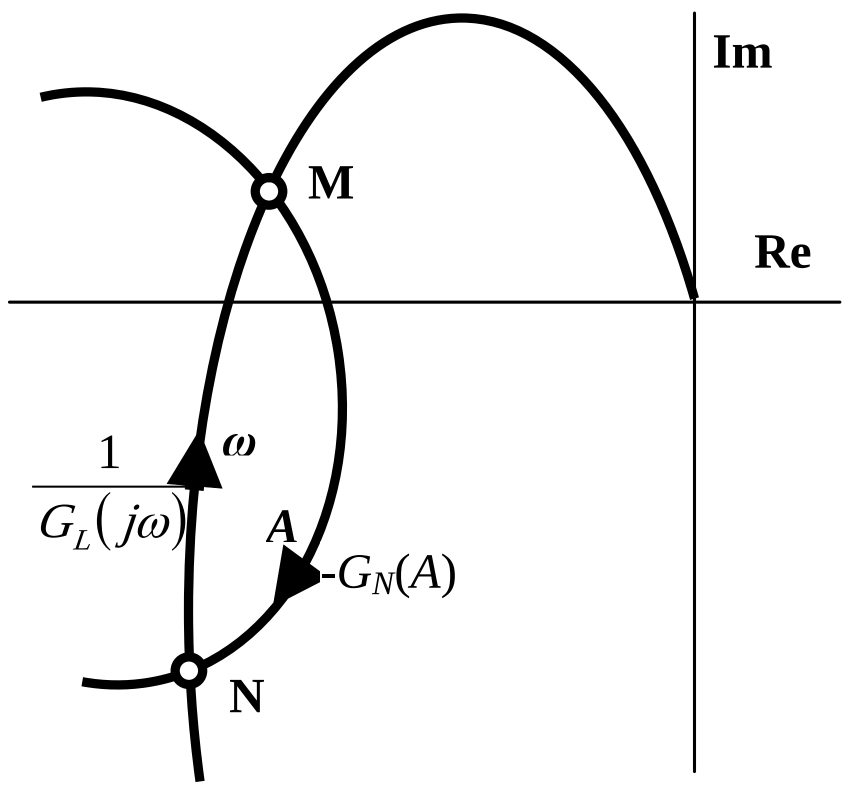
⇩

real solution

⇩

autooscillations exist and have computed parameters

Graphic solution :



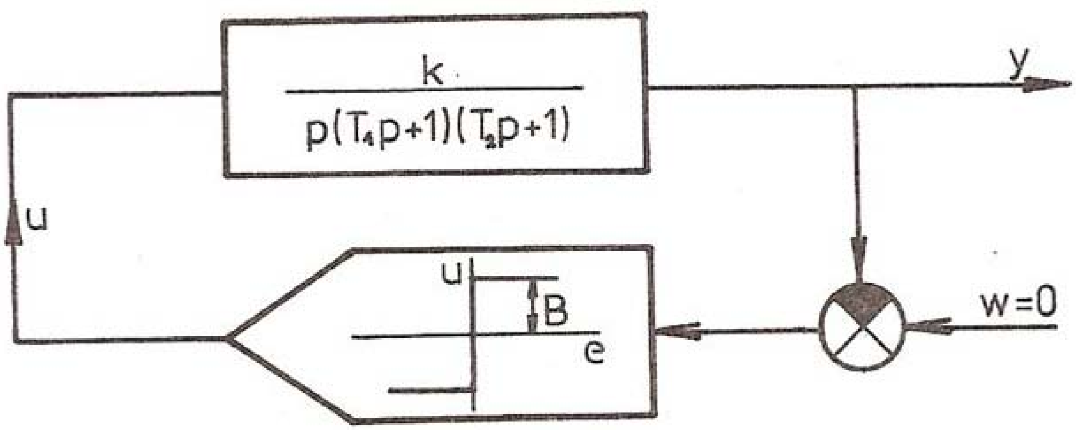
$$G_L(j\omega) = -\frac{1}{G_N(A)}$$

$$-G_L(j\omega) = \frac{1}{G_N(A)}$$

$$\frac{1}{G_L(j\omega)} = -G_N(A)$$

Intersections M , N
? exist – not exist ?

Example: Nonlinear control circuit in figure consists of linear proportional system



(k = 1; T_1 = 1; T_2 = 1) with ideal two-value controller (B=1). Determine if in the circuit emerge autooscillations and if they do determine amplitude and frequency.

Solve analytically and graphically.

Solution:
Analytic:

$$G_N(A)=\frac{4B}{\pi A}\sqrt{1-\frac{b^2}{A^2}} \quad \rightarrow \quad b=0 \qquad G_N(A)=\frac{4B}{\pi A}$$

$$G_L(j\omega)G_N(A)+1=0 \quad \rightarrow \quad \frac{k}{j\omega(T_1 j\omega+1)(T_2 j\omega+1)}\cdot\frac{4B}{\pi A}+1=0$$

$$k\frac{4B}{\pi A}-(T_1+T_2)\omega^2+j\omega\left(1-T_1T_2\omega^2\right)=0$$

$$\mathrm{Re}=0: \qquad k\frac{4B}{\pi A}-(T_1+T_2)\omega^2=0$$

$$\mathrm{Im}=0: \qquad \omega\left(1-T_1T_2\omega^2\right)=0$$

Solution exists, autooscillations emerge. Amplitude and frequency:

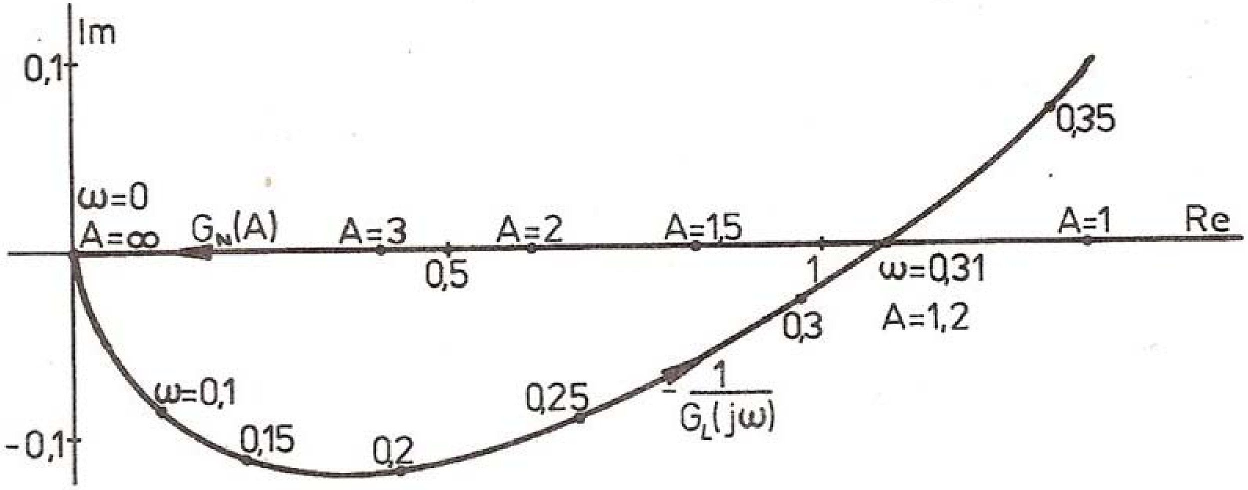
$$\omega=\sqrt{\frac{1}{T_1T_2}}=\sqrt{\frac{1}{1.10}}=0{,}316$$

$$A=\frac{T_1T_2}{T_1+T_2}\cdot\frac{4Bk}{\pi}=\frac{1.10}{1+10}\cdot\frac{4.1.1}{\pi}=1{,}1575$$

Graphic:

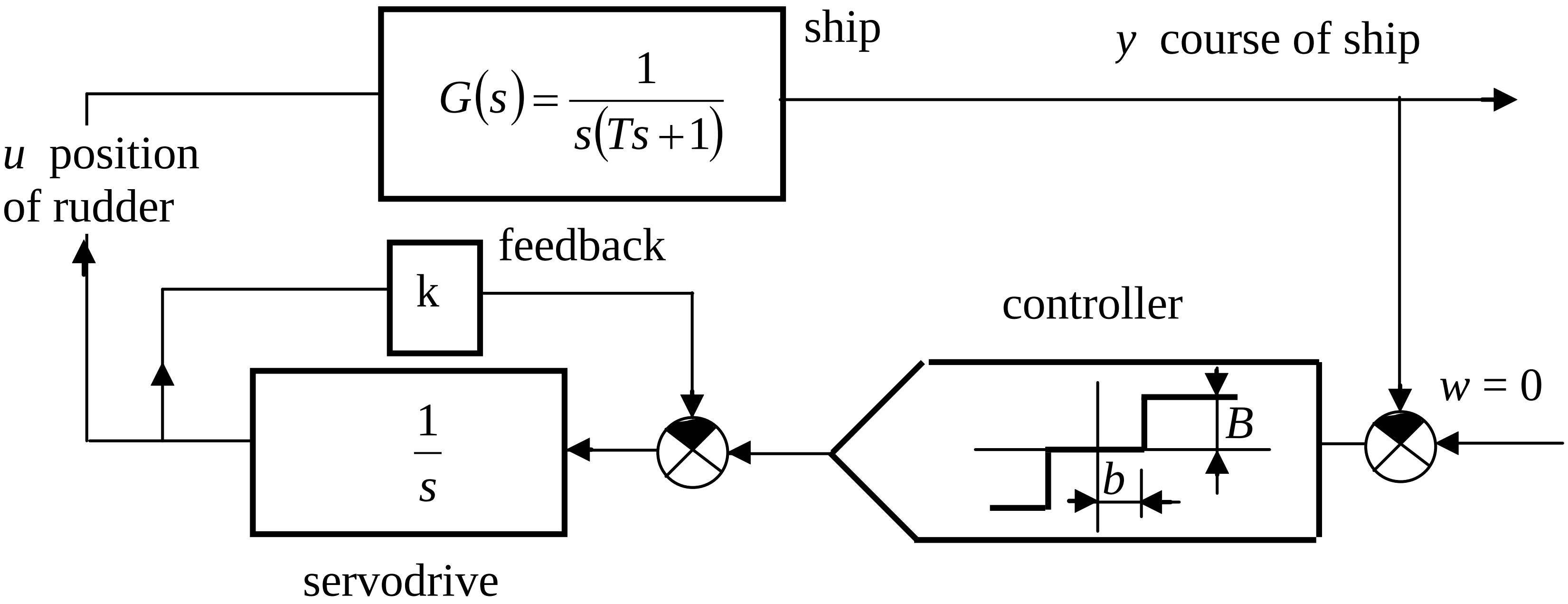
$$G_N(A) = -\frac{1}{G_L(j\omega)} \quad \rightarrow \quad G_N(A) = \frac{4B}{\pi A}$$

$$-\frac{1}{G_L(j\omega)} = -\frac{1}{k} j\omega(T_1 j\omega + 1)(T_2 j\omega + 1)$$



$$\omega = 0{,}31; \quad A = 1{,}2$$

Example: Course of ship is controlled using three-value controller. Scheme of controller circuit in figure. Time constant of the ship as controlled system is T = 100 s and gain k = 10. Determine when the autooscillations do not emerge.

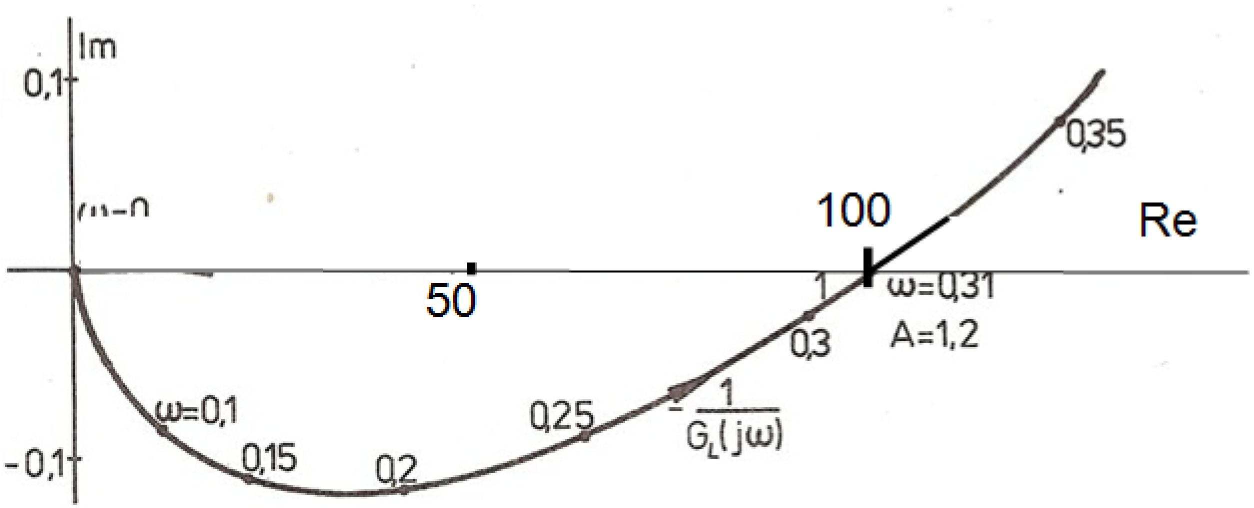


Solution: Linear member

$$G(s)=\frac{1}{s(Ts+1)(s+k)}$$

Condition for autooscillation emergence

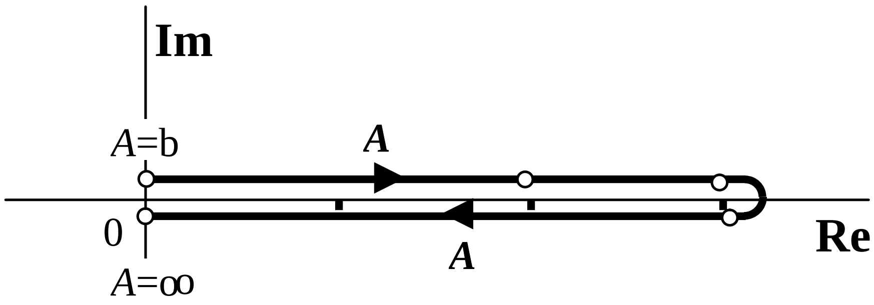
$$G_N(A)=-\frac{1}{G_L(j\omega)}$$



Equivalent transfer of nonlinear member and it's amplitude characteristics.

$$G_N(A) = \frac{4B}{\pi A}\sqrt{1 - \frac{b^2}{A^2}}$$

Value *A* for which is maximum of characteristic.



For derivative use these formulas:

$$y = uv \qquad y' = u'v + uv' \qquad y = \frac{1}{x^2} \qquad y' = -\frac{2}{x^3}$$

$$y = \frac{1}{x} \qquad y' = -\frac{1}{x^2} \qquad y = \sqrt{x} \qquad y' = \frac{1}{2\sqrt{x}}$$

$$\frac{dG_N(A)}{dA} = 0$$

Derivative $\frac{dG_N(A)}{dA}=0$

$$\left(\frac{4B}{\pi A}\sqrt{1-\frac{b^2}{A^2}}\right)'=-\frac{4B}{\pi A^2}\sqrt{1-\frac{b^2}{A^2}}+\frac{4B}{\pi A}\frac{1}{2\sqrt{1-\frac{b^2}{A^2}}}\frac{2b^2}{A^3}=0$$

$$-\sqrt{1-\frac{b^2}{A^2}}+\frac{b^2}{A^2}\frac{1}{\sqrt{1-\frac{b^2}{A^2}}}=0$$

$$-\left(1-\frac{b^2}{A^2}\right)+\frac{b^2}{A^2}=0 \quad\Rightarrow\quad -1+2\frac{b^2}{A^2}=0 \quad\Rightarrow\quad \frac{b^2}{A^2}=\frac{1}{2}$$

$$\frac{b}{A}=\frac{1}{\sqrt{2}} \quad\Rightarrow\quad A=b\sqrt{2}$$

So the characteristics will not intersect (autooscillations will not emerge).

$$G_N(A)=\frac{4B}{\pi A}\sqrt{1-\frac{b^2}{A^2}}=\frac{4B}{\pi b\sqrt{2}}\sqrt{1-\frac{b^2}{b^2.2}}<100$$